SYMMETRIC TENSORS OF THE SECOND ORDER WHOSE FIRST COVARIANT DERIVATIVES ARE ZERO*

BY

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1. Consider a Riemann space of the nth order, whose fundamental quadratic form, assumed to be positive definite, is written

$$ds^2 = g_{rs}\, dx^r\, dx^s \qquad (g_{rs} = g_{sr}), \tag{1}$$

where r and s are summed from 1 to n in accordance with the usual convention which will be followed throughout this paper. It is well known that the first covariant derivatives $g_{rs/t}$ are zero, where

$$g_{rs/t} = \frac{\partial g_{rs}}{\partial x^t} - g_{ra}\, \Gamma^a_s - g_{as}\, \Gamma^a_{rt} \tag{2}$$

and

$$\Gamma^a_{st} = \tfrac{1}{2}\, g^{ap} \left(\frac{\partial g_{sp}}{\partial x^t} + \frac{\partial g_{tp}}{\partial x^s} - \frac{\partial g_{st}}{\partial x^p} \right), \tag{3}$$

the function g^{ap} being the cofactor of g_{ap} in the determinant

$$g = |g_{rs}| \tag{4}$$

divided by g. It is the purpose of this paper to determine the necessary and sufficient conditions that there exist a symmetric covariant tensor a_{rs} such that the first covariant derivatives $a_{rs/t}$ are zero, or more than one such tensor.

2. Let a_{rs} denote the covariant components of any symmetric tensor of the second order. If ϱ_h is a root of the equation

$$|a_{rs} - \varrho\, g_{rs}| = 0, \tag{5}$$

the functions λ^r_h $(r = 1, \ldots, n)$ defined by

$$(a_{rs} - \varrho_h\, g_{rs})\, \lambda^r_h = 0 \qquad (s = 1, \ldots, n) \tag{6}$$

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are the contravariant components of a vector. It is well known that the roots of (5) are real, and that if they are simple, the n corresponding vectors at a point are mutually orthogonal.* Moreover, if a root is of order m, equations (6) admit m sets of independent solutions, and any linear combination of them is also a solution. It is possible to choose m solutions so that the corresponding vectors at a point are mutually orthogonal, and thus from (6) obtain n sets of solutions so that the corresponding vectors at a point are orthogonal; that is,

$$g_{rs}\lambda_h^r \lambda_k^s = 0 \qquad (h, k = 1, \ldots, n;\ h \neq k). \tag{7}$$

Moreover, the components may be chosen so that

$$g_{rs}\lambda_h^r \lambda_h^s = 1 \qquad (h = 1, \ldots, n), \tag{8}$$

that is, the vectors are unit vectors.

The curves in space whose direction at each point is defined by λ_h^r form a congruence of curves C_h. Thus equations (6) define an n-uple of congruences of curves, such that the curves of the n-uple through a point are mutually orthogonal.

The covariant components $\lambda_{h,r}$ of the vector h are given by

$$\lambda_{h,r} = g_{rs}\lambda_h^s, \qquad \lambda_h^s = g^{rs}\lambda_{h,r}, \tag{9}$$

and hence (7) and (8) are equivalent to

$$\lambda_{h,r}\lambda_k^r = \delta_{hk}, \tag{10}$$

where

$$\delta_{hk} = 1 \text{ for } h = k; = 0 \text{ for } h \neq k. \tag{11}$$

The functions γ_{hij} defined by

$$\gamma_{hij} = \lambda_{h,r/s}\lambda_i^r \lambda_j^s, \tag{12}$$

where $\lambda_{h,r/s}$ is the covariant derivative of $\lambda_{h,r}$ with respect to x^s, are invariants; they are called *rotations* by Ricci and Levi-Civita.† They have shown that

$$\gamma_{hij} + \gamma_{ihj} = 0, \qquad \gamma_{hhi} = 0 \qquad (h, i, j = 1, \ldots, n). \tag{13}$$

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* Cf. these Transactions, vol. 25 (1923), p. 259.

† Mathematische Annalen, vol. 54 (1901), p. 148; also, Wright, *Invariants of Quadratic Differential Forms*, Cambridge Tract, No. 9, p. 68.

From (12) we have

$$\lambda_{h,r/s} = \sum_{i,j}^{1\ldots n} \gamma_{hij}\, \lambda_{i,r}\, \lambda_{j,s}, \tag{14}$$

and since $g_{rs/t} = 0$, it follows from (9) that

$$\lambda^p_{h/s} = \sum_{i,j}^{1\ldots n} \gamma_{hij}\, \lambda^p_i\, \lambda_{j,s}. \tag{15}$$

3. If all the roots of (5) are equal, we must have $\alpha_{rs} = \varrho\, g_{rs}$. Differentiating covariantly with respect to x^t, and making use of the fact that $g_{rs/t} = 0$ and the assumption that $\alpha_{rs/t} = 0$, we have that ϱ is constant. Consequently α_{rs} is essentially the same as g_{rs}. We exclude this case from further consideration.

Since (7) is satisfied whether the functions λ^r_h and λ^r_k correspond to different simple roots of (5), or to the same multiple root when such exists, we have from (6)

$$\alpha_{rs}\, \lambda^r_h\, \lambda^s_k = 0 \qquad (h, k = 1, \ldots, n;\ h \neq k). \tag{16}$$

Also from (6) we have

$$\alpha_{rs}\, \lambda^r_h\, \lambda^s_h = \varrho_h. \tag{17}$$

From (17) we have by differentiating covariantly with respect to x^t and making use of (15), (16), and (17)

$$\alpha_{rs/t}\, \lambda^r_h\, \lambda^s_h = \frac{\partial \varrho_h}{\partial x^t}. \tag{18}$$

Also from (16) we have, because of (13), (14), (16) and (17),

$$\alpha_{rs/t}\, \lambda^r_h\, \lambda^s_k + \sum_{j}^{1\ldots n} (\varrho_k - \varrho_h)\, \gamma_{hkj}\, \lambda_{j,t} = 0.$$

Multiplying by λ^t_l and summing for t, we have

$$\alpha_{rs/t}\, \lambda^r_h\, \lambda^s_k\, \lambda^t_l + (\varrho_k - \varrho_h)\, \gamma_{hkl} = 0 \qquad (h \neq k). \tag{19}$$

From (18) it follows that if $\alpha_{rs/t} = 0$ the roots ϱ are constant. And from (19) we have for two different roots

$$\gamma_{hkl} = 0 \qquad (h \neq k). \tag{20}$$

Let ϱ_1 be a root of (5) which we assume to be a multiple root of order m, and denote by λ^r_h $(h = 1, \ldots, m)$ the components of the m mutually orthogonal vectors corresponding to it, and by λ^r_k $(k = m+1, \ldots, n)$ the components of the directions corresponding to the other roots of (5). From (20) we have

$$\gamma_{hkl} = 0 \qquad (h = 1, \ldots, m;\ k = m+1, \ldots, n;\ l = 1, \ldots, n). \tag{21}$$

Consider the system of equations

$$X_k(f) \equiv \lambda^r_k \frac{\partial f}{\partial x^r} = 0 \qquad (k = m+1, \ldots, n). \tag{22}$$

If we introduce the notation

$$\frac{\partial f}{\partial s^k} = \lambda^r_k \frac{\partial f}{\partial x^r},$$

then, as Ricci and Levi-Civita have shown*, the relation

$$\frac{\partial}{\partial s_j}\frac{\partial f}{\partial s_k} - \frac{\partial}{\partial s_k}\frac{\partial f}{\partial s_j} = \sum_i^{1\ldots n} (\gamma_{ijk} - \gamma_{ikj}) \frac{\partial f}{\partial s_i} \tag{23}$$

is satisfied for any function f.

Applying this formula to equations (22) we have in consequence of (21)

$$X_j X_k(f) - X_k X_j(f) = \sum_i^{m+1\ldots n} (\gamma_{ijk} - \gamma_{ikj}) X_i(f) \qquad (j, k = m+1, \ldots, n).$$

Hence the system (22) is complete and admits m independent solutions, say f_h $(h = 1, \ldots, m)$.

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* Loc. cit., p. 150; Wright, p. 69.

Let ρ_2 be another root of (5), of order p, and denote by λ_j^r $(j = m+1, \ldots, m+p)$ the components of the corresponding vectors. In like manner we show that the equations

$$\lambda_l^r \frac{\partial f}{\partial x^r} = 0 \qquad (l = 1, \ldots, m, m+p+1, \ldots, n)$$

form a complete system and admit p independent solutions f_j $(j = m+1, \ldots, m+p)$.

From (22) and the equations

$$\lambda_k^r \lambda_{h,r} = 0 \qquad (h = 1, \ldots, m;\ k = m+1, \ldots, n)$$

it follows that there exist functions a_h^σ such that

$$\frac{\partial f_h}{\partial x^r} = \sum_\sigma a_h^\sigma \lambda_{\sigma,r} \qquad (h, \sigma = 1, \ldots, m).$$

In like manner, we have

$$\frac{\partial f_j}{\partial x^r} = \sum_\tau b_j^\tau \lambda_{\tau,r} \qquad (j, \tau = m+1, \ldots, m+p).$$

Consequently we have

$$g^{rs} \frac{\partial f_h}{\partial x^r} \frac{\partial f_j}{\partial x^s} = \sum_{\sigma,\tau} a_h^\sigma b_j^\tau g^{rs} \lambda_{\sigma,r} \lambda_{\tau,s} = 0,$$

that is, any hypersurface $f_h =$ const. is orthogonal to each of the hypersurfaces $f_j =$ const.

Proceeding in this manner with the other roots of (5) we obtain a group of hypersurfaces corresponding to each distinct root of (5), the number of hypersurfaces in a group being equal to the order of the root. Any two hypersurfaces of two different groups are orthogonal to one another. If we take these n families of hypersurfaces for the parametric surfaces $x^r =$ const. $(r = 1, \ldots, n)$, it follows that the functions g_{rs} are zero, for the case where $x^r =$ const. and $x^s =$ const. are hypersurfaces of different groups; in this sense we say that r and s refer to different groups, or different roots of (5).

From the equations (22) for this choice of the variables x, it follows that $\lambda_k^r = 0$, for r and k referring to different roots of (5). From (9) it follows also that $\lambda_{k,r} = 0$ for k and r referring to different roots.

Equations (6) may be replaced by*

$$\alpha_{rs} = \sum_{h}^{1\ldots n} \varrho_h \, \lambda_{h,r} \, \lambda_{h,s} \tag{24}$$

whether the roots of (5) are simple, or some are multiple. From (24) and the preceding observations it follows

$$\begin{aligned} \alpha_{rs'} &= g_{rs'} = 0, \\ \alpha_{rs} &= \varrho_h \, g_{rs}, \end{aligned} \tag{25}$$

where r and s' refer to any two different roots and r and s refer to the root ϱ_h.†

4. From (25) we have $\alpha_{rs'} = 0$, hence if $\alpha_{rs'/t} = 0$, we must have (cf. (2))

$$\alpha_{rl} \, \Gamma^{l}_{s't} + \alpha_{s'q} \, \Gamma^{q}_{rt} = 0 \qquad (l, q = 1, \ldots, n),$$

that is

$$\alpha_{rl} \, g^{l\beta} \left[\frac{\partial g_{s'\beta}}{\partial x^t} + \frac{\partial g_{t\beta}}{\partial x^{s'}} - \frac{\partial g_{s't}}{\partial x^\beta} \right] + \alpha_{s'q} \, g^{q\beta} \left[\frac{\partial g_{r\beta}}{\partial x^t} + \frac{\partial g_{t\beta}}{\partial x^r} - \frac{\partial g_{rt}}{\partial x^\beta} \right] = 0.$$

If r refers to the root ϱ_1 of (5), say $r = 1, \ldots, m$ and s' to the root ϱ_2, say $s' = m + 1, \ldots, m + p$, we have from (25)

$$\begin{aligned} \alpha_{rl} &= \varrho_1 \, g_{rl} && (l = 1, \ldots, m); \\ \alpha_{rl} &= 0 && (l = m + 1, \ldots, n); \\ \alpha_{s'q} &= \varrho_2 \, g_{s'q} && (q = m + 1, \ldots, m + p); \\ \alpha_{s'q} &= 0 && (q = 1, \ldots, m, m + p + 1, \ldots, n). \end{aligned}$$

Hence the above equation reduces to

$$\varrho_1 \left(\frac{\partial g_{s'r}}{\partial x^t} + \frac{\partial g_{tr}}{\partial x^{s'}} - \frac{\partial g_{s't}}{\partial x^r} \right) + \varrho_2 \left(\frac{\partial g_{rs'}}{\partial x^t} + \frac{\partial g_{ts'}}{\partial x^r} - \frac{\partial g_{rt}}{\partial x^{s'}} \right) = 0.$$

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* Cf. Ricci and Levi-Civita, loc. cit., p. 159.

† Cf. Levi-Civita, Annali di Matematica, ser. 2, vol. 24 (1896), p. 298.

If now t and r refer to the same root, this equation reduces to

$$(\varrho_1 - \varrho_2)\frac{\partial g_{tr}}{\partial x'^s} = 0,$$

and if t and s' refer to the same root, we have

$$(\varrho_1 - \varrho_2)\frac{\partial g_{s't}}{\partial x^r} = 0.$$

If r, s' and t refer to three different roots, the equation vanishes identically.

Since ϱ_1 and ϱ_2 are not equal by hypothesis, we have that each function g_{rs} depends only on the coördinates referring to the same root as r and s.

Consider again

$$\alpha_{rs} = \varrho_1 g_{rs} \qquad (r, s = 1, \ldots, m).$$

Now

$$\alpha_{rs/t} = \varrho_1 \frac{\partial g_{rs}}{\partial x^t} - \alpha_{rl}\,\Gamma^l_{st} - \alpha_{sl}\,\Gamma^l_{rt} \qquad (l = 1, \ldots, n),$$

which by (25) is reducible to

$$\alpha_{rs/t} = \varrho_1\left(\frac{\partial g_{rs}}{\partial x^t} - g_{rl}\,\Gamma^l_{st} - g_{sl}\,\Gamma^l_{rt}\right) = \varrho_1\, g_{rs/t} = 0.$$

Hence we have the following theorem:

A necessary and sufficient condition that a Riemann space admit a symmetric covariant tensor of the second order α_{rs} other than, with a positive definite fundamental form (1), *g_{rs}, such that its first covariant derivative is zero, is that* (1) *be reducible to a sum of forms*

$$\varphi^{(i)} = g^{(i)}_{r_i s_i}\, dx^{r_i}\, dx^{s_i}, \tag{26}$$

where $g^{(i)}_{r_i s_i}$ are functions at most of the x's of that form; *then*

$$\alpha_{rs}\, dx^r\, dx^s = \sum_i \varrho_i\, \varphi^{(i)}, \tag{27}$$

where the ϱ's are arbitrary constants.

In particular, if all the roots of (5) are simple, the space is euclidean; if its fundamental form is taken in the form

$$ds^2 = \sum_i dx^{i^2} \qquad (i = 1, \ldots, n),$$

then

$$a_{rs}\, dx^r\, dx^s = \sum_i \varrho_i\, dx^{i^2} \qquad (i = 1, \ldots, n),$$

where the ϱ's are n different arbitrary constants.

When any one of the roots of (5) is simple, the corresponding congruence is normal, and the tangents to the congruence form a field of parallel vectors in the sense of Levi-Civita.*

5. In this section it will be shown that the problem of determining whether a given Riemann space admits one, or more, symmetric tensors whose first covariant derivatives are zero is a problem of algebra.†

We recall that if a_{rs} is any symmetric tensor, then

$$a_{rs/jk} - a_{rs/kj} = a_{rt}\, B^t_{sjk} + a_{st}\, B^t_{rjk},$$

where $a_{rs/jk}$ is the second covariant derivative of a_{rs}, and B^t_{sjk} are the components of the Riemann tensor of the second kind formed with respect to (1). If then $a_{rs/j} = 0$, we must have

$$a_{rs/jk} - a_{rs/kj} = 0, \tag{28}$$

and consequently we have equations of the form

$$a_{rt}\, B^t_{sjk} + a_{st}\, B^t_{rjk} = 0 \qquad (j, k, r, s, t = 1, \ldots, n). \tag{29}$$

Differentiating these equations covariantly successively we have the sets of equations

$$\begin{aligned} &a_{rt}\, B^t_{sjk/m_1} + a_{st}\, B^t_{rjk/m_1} = 0, \\ &\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots \\ &a_{rt}\, B^t_{sjk/m_1\ldots m_l} + a_{st}\, B^t_{rjk/m_1\ldots m_l} = 0, \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{30}$$

* Cf. Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 211.

† Cf. Eisenhart and Veblen, Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 23; also Veblen and Thomas, these Transactions, vol. 25 (1923).

Since g_{rs} satisfies (28) the systems (29) and (30) are satisfied by g_{rs} and consequently are algebraically consistent. From this it follows either that the functions g_{rs} are the only solution of (29) and (30), or that (29) and the first $l\,(\geqq 0)$ sets of (30) admit a complete system of solutions g_{rs} and $\overset{(1)}{\alpha}_{rs}, \ldots, \overset{(p)}{\alpha}_{rs}$ which satisfy also the $(l+1)$th set of equations (30). In the latter case the general solution is of the form

$$\alpha_{rs} = \overset{(0)}{\varphi}\, g_{rs} + \overset{(1)}{\varphi}\, \overset{(1)}{\alpha}_{rs} + \cdots + \overset{(p)}{\varphi}\, \overset{(p)}{\alpha}_{rs}. \tag{31}$$

If any one of the functions $\overset{(\sigma)}{\alpha}_{rs}\ (\sigma = 1, \ldots, p)$ is substituted in (29) and the first l sets of (30), and these equations are differentiated covariantly, we have, in consequence of the above requirement, that the functions $\overset{(\sigma)}{\alpha}_{rs/m}$ $(\sigma = 1, \ldots, p;\ m = 1, \ldots, n)$ satisfy (29) and the first l sets of (30). Consequently we have

$$\overset{(\sigma)}{\alpha}_{rs/m} = \lambda^{(\sigma 0)}_{m}\, g_{rs} + \lambda^{(\sigma 1)}_{m}\, \overset{(1)}{\alpha}_{rs} + \cdots + \lambda^{(\sigma p)}_{m}\, \overset{(p)}{\alpha}_{rs}, \tag{32}$$

where the $p(p+1)$ vectors $\lambda^{(\sigma\beta)}_{m}\ (\sigma = 1, \ldots, p;\ \beta = 0, 1, \ldots, p)$ must be such that the functions (32) shall satisfy (28). Substituting in these equations we find that the functions λ must satisfy the system

$$\frac{\partial \lambda^{(\sigma\tau)}_{p}}{\partial x^{q}} - \frac{\partial \lambda^{(\sigma\tau)}_{q}}{\partial x^{p}} + \sum_{\omega} \left(\lambda^{(\sigma\omega)}_{p}\, \lambda^{(\omega\tau)}_{q} - \lambda^{(\sigma\omega)}_{q}\, \lambda^{(\omega\tau)}_{p} \right) = 0 \quad \left(\begin{matrix} \sigma, \omega = 1, \ldots, p \\ \tau = 0, 1, \ldots, p \end{matrix} \right). \tag{33}$$

In order that α_{rs} given by (31) shall satisfy $\alpha_{rs/t} = 0$, it is necessary and sufficient that the functions $\varphi^{(i)}$ satisfy

$$\frac{\partial \varphi^{(0)}}{\partial x^{t}} + \sum_{\sigma} \varphi^{(\sigma)}\, \lambda^{(\sigma 0)}_{t} = 0 \qquad (\sigma = 1, \ldots, p), \tag{34}$$

and

$$\frac{\partial \varphi^{(\tau)}}{\partial x^{t}} + \sum_{\sigma} \varphi^{(\sigma)}\, \lambda^{(\sigma\tau)}_{t} = 0 \qquad (\sigma, \tau = 1, \ldots, p). \tag{35}$$

In consequence of (33) equations (35) are completely integrable and therefore admit solutions involving p arbitrary constants. Because of (33) the conditions of integrability of (34) are satisfied; hence $\varphi^{(0)}$ involves these p arbitrary constants and an additive arbitrary constant which may be neglected.*

* If a_{rs} is a tensor whose first covariant derivative is zero, so also is $a_{rs} + \lambda g_{rs}$, where λ is an arbitrary constant.

In view of the above results we have the theorem:

If equations (29) *and the first* $l\,(\geqq 0)$ *sets of equations* (30) *admit a complete system of solutions* g_{rs} *and* $a_{rs}^{(\sigma)}$ $(\sigma = 1, \ldots, p)$ *which are also solutions of the* $(l+1)$*th set of equations* (30), *there exists a symmetric tensor of the second order, involving p arbitrary constants, whose first covariant derivative is zero.*

6. Suppose that the fundamental form is the sum of j forms (26). By definition

$$B^{a}_{prs} = g^{aq} B_{pqrs}, \tag{36}$$

where B_{pqrs} is the covariant Riemann tensor of the fourth order, that is,

$$\begin{aligned} B_{pqrs} = \frac{1}{2}\left(\frac{\partial^2 g_{ps}}{\partial x^q\,\partial x^r} + \frac{\partial^2 g_{qr}}{\partial x^p\,\partial x^s} - \frac{\partial^2 g_{pr}}{\partial x^q\,\partial x^s} - \frac{\partial^2 g_{qs}}{\partial x^p\,\partial x^r}\right) \\ + g^{lm}\left(\Gamma_{ps,m}\,\Gamma_{qr,l} - \Gamma_{pr,m}\,\Gamma_{qs,l}\right), \end{aligned} \tag{37}$$

where

$$\Gamma_{ps,m} = \frac{1}{2}\left(\frac{\partial g_{pm}}{\partial x^s} + \frac{\partial g_{sm}}{\partial x^p} - \frac{\partial g_{ps}}{\partial x^m}\right). \tag{38}$$

For the case under consideration, namely (26), it is readily shown that the components B_{pqrs} are zero, unless p, q, r, s refer to the same root of (5); likewise B^{a}_{prs}, and its first covariant derivatives $B^{a}_{prs|t}$. Consequently equations (29) and the first set of (30) admit, in addition to g_{rs}, the j sets of solutions of the form (25). If it is understood that each of the forms (26) is not further reducible to sums of such forms, we have a complete set of solutions of (29). Hence when the space is referred to the coördinates giving (25) the number l in the preceding theorem is zero.

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